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Component mode synthesis (CMS) based on an enriched ritz approach for efficient structural optimization

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Abstract

This paper presents a new method to improve the standard superelement reduction methods by taking into account a priori information concerning the potential structural modifications, thus allowing the same transformation to be used throughout the optimization process. The aim is to create a single enriched model reduction transformation that preserves the precision of the reduced substructure model throughout the optimization process. The approach consists firstly in extending the standard condensation basis by a set of optimized static residual and secondly, in eliminating the associate coordinates from the reduced system. The proposed method can be used with any condensation procedure, including both direct reductions and component mode synthesis approaches with any kind of substructure natural modes; free–free, clamped or hybrid modes.

1. Introduction

The ever increasing demand for faster engineering analysis in the design process has resulted in a substantial amount of research and development on faster and more accurate approximate reanalysis method. Over the past 20 years, the modeling of complex industrial structures lead to finite element models of very large sizes (for example an aerospace structure can be composed by more than 10^6 dofs). Nevertheless, many details of such large models are not critical when it comes to capturing the input/output relationship. It is thus important to reduce the size of the system for mainly several reasons: the cost of computation when one wants to extract the eigensolutions or to predict the behavior of the full structure, the optimization or updating procedures which require fast iterative techniques and finally nonlinear mechanics.

The difficulty of any model reduction procedure lies on how to complete the representation basis in order to reduce truncation effects. Two main classes of methodologies can be found in the literature. The first class seeks to generate a set of Ritz vectors capable of representing with precision the structural behavior under a wide variety of structural modifications. For example, Balmès [1,2], Guillaume et al. [3] studied the possibility of using a constant Ritz basis to create parametric families of reduced models. Bouazzouni et al. [4,5] developed the other side an optimal method to construct additional vectors by using the dynamic behavior of

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the structure before modification combined with the a priori knowledge of the design variables. Both of these approaches have already been used effectively in an industrial context. The second class of reduction methods are based on a high-order polynomial expansions of model responses about a nominal point in parameter space. This approach proves to be particularly interesting for small number of design variables and can be used for topological optimization.

Moreover, the substructuring or also known as component mode synthesis (CMS) method currently plays a considerable role in the analysis of the complex structures. This method consists in subdividing a structure in components called substructure or superelement, which are analyzed and condensed separately, while preserving the junction dofs between substructures. The substructures, or components, are represented by their modes (in the sense of Ritz vectors), the latter include the vibration normal modes, the rigid body modes, the static modes, the interface modes, etc. The advantages of such a method are to be able to analyze problems of quasi-unlimited size with the current data-processing resources. The structural modification of a substructure makes it possible to reanalyze the full structure with a low cost of computation and it makes it possible to prepare each model of substructures independently of the others. This advantage is significant for the study of the complex substructures whose designs are entrusted to various equipment suppliers or in the context of parallel processing. Indeed different computers can carry out the preparation of each reduced model at the same time. Several CMS methods have been developed for 40 years. All of them make use of the vibration normal modes of the substructures. Depending on the boundary conditions applied to the substructure interfaces when these normal modes are obtained, the CMS methods can be classified into four groups: fixed interface methods [6,7] free interface methods [8–13], hybrid interface methods [8], loaded interface methods [14]. The variants in each group differ from each other mainly in the choice of the supplementary Ritz vectors and the associated generalized coordinates, and in the coupling procedure.

However, these methods are not always well adapted to the industrial problem. This is especially true when parametric studies are to be performed with respect to design variables contained within individual superelements. The designer has the choice of either reusing the nominal model reduction transformation or performing a new superelement analysis for each modified component. The first option generally leads to inaccurate results while the latter is often impracticable due to cost considerations.

We propose in this paper a new method to improve the standard superelement reduction methods by taking into account a priori knowledge of the potentially modifiable design variables with the objective of constructing a unique reduction transformation matrix which will preserve the predictiveness of the superelement for a wide range of structural modifications. The approach consists firstly, in extending the standard condensation basis by a set of optimized static residuals and secondly, in eliminating the associate coordinates from the reduced system. The global parameterized model is thus constituted of an assembly of individually reduced and parameterized component models. The potential of the proposed methodology is demonstrated on the basis of an academic simulated numerical test case and a numerical test case taken from the aero-engine industry.

2. Background

2.1. Standard CMS

The objective of CMS is to calculate the dynamic behavior of a structure, starting from the knowledge of the dynamic behavior of its substructures when they are described by numerical models of reduced size. The general principle of substructuring consists in approximating the dynamic behavior of a substructure by a linear combination of q vectors that form a Ritz basis denoted by the matrix $\mathbf{T}$. The size of the model is reduced from N physical dofs to q "privileged" or generalized dofs. The CMS matrix $\mathbf{T}$ discussed in this section may or may not be identical to the condensation matrix of a direct reduction method. Differences between the substructuring methods are based upon: (1) the choice of reduction basis $\mathbf{T}$ and generalized dof vector c_{I} (in the remainder, c_{I} denotes the vector of generalized dofs defined across the interface common to the N_S substructures); (2) the technique of assembly of the substructures.

The basic philosophy behind the creation of a superelement from the global stiffness and mass matrices of a structural component is briefly described below in the context of a linear conservative elastodynamic substructure. The homogeneous equation of motion for such a substructure is given by

$$(\mathbf{K} - \lambda\mathbf{M})\mathbf{y} = \{\mathbf{f}\}, \tag{1}$$

where **M**, **K** represent the discrete mass and stiffness matrices of the system and **y**, γ are a given pair of eigensolutions and $\{\mathbf{f}\} = \{\mathbf{f}_e + \mathbf{f}_j\}$ where $\mathbf{f}_e$ is the vector of external forces and $\mathbf{f}_j$ are the interface reactions. The generic substructure model reduction is performed via a transformation matrix **T**:

$$\mathbf{y} = \mathbf{Tq}, \tag{2}$$

where **q** is a vector of generalized coordinates representing the contribution of each column of **T**. Note that some of these coordinates may correspond to physical degrees of freedom.

Substituting Eq. (2) into Eq. (1) and premultiplying by the transpose of **T** yields

$$\left(\hat{\mathbf{K}} - \lambda\hat{\mathbf{M}}\right)\mathbf{q} = \mathbf{T}^{\mathrm{T}}\{\mathbf{f}\}, \tag{3}$$

where $\hat{\mathbf{K}} = \mathbf{T}^{\mathrm{T}}\mathbf{KT}$ and $\hat{\mathbf{M}} = \mathbf{T}^{\mathrm{T}}\mathbf{MT}$ are the reduced stiffness and mass matrices.

The dofs of the displacement vector $\mathbf{y}_s$ of the substructure are partitioned into subsets of junction dofs $\mathbf{y}_j$ and interior dofs $\mathbf{y}_i$:

$$\{\mathbf{y}_s\} = \begin{Bmatrix} \mathbf{y}_j \\ \mathbf{y}_i \end{Bmatrix}. \tag{4}$$

The partitioned stiffness and mass matrices of the substructure inherit the following forms:

$$\mathbf{K}_s = \begin{bmatrix} \mathbf{K}_{jj} & \mathbf{K}_{ji} \\ \mathbf{K}_{ij} & \mathbf{K}_{ii} \end{bmatrix} \quad \text{and} \quad \mathbf{M}_s = \begin{bmatrix} \mathbf{M}_{jj} & \mathbf{M}_{ji} \\ \mathbf{M}_{ij} & \mathbf{M}_{ii} \end{bmatrix}. \tag{5}$$

The transformation **T** is a Ritz basis classically constituted by a combination of static responses (constraints modes, attachment modes) and normal modes (fixed interfaces modes [6,7], free interfaces modes [8–13], loaded interfaces modes [14]).

2.2. Example of a CMS method: Craig– Bampton method

In dynamic analysis of complex structures with a large number of dof's the Craig–Bampton method is a well-known method to drastically reduce the overall number of dof's. This method combines motion of interface (constraint modes) with modes of the substructure assuming the interface is held fixed (normal modes). The Craig–Bampton transformation (CBT) for a substructure is defined as

$$\{\mathbf{y}_s\} = \begin{bmatrix} \mathbf{y}_j \\ \mathbf{y}_i \end{bmatrix} = \begin{bmatrix} \mathbf{I}_{jj} & 0 \\ \mathbf{G}_{ij} & \Phi \end{bmatrix} \begin{bmatrix} \mathbf{y}_j \\ \mathbf{c} \end{bmatrix}, \tag{6}$$

where G_{ij} is the static transformation matrix (Guyan matrix) and $\Phi = [j_1 \ldots j_{nc}]$ is the normal modes matrix with fixed interface.

The reduced problem is obtained by substituting Eq. (6) into Eq. (1) and premultiplying by the transpose of the transformation matrix. The reduced stiffness and mass matrices are expressed as

$$\hat{\mathbf{K}} = \begin{bmatrix} \mathbf{K}_j & 0 \\ 0 & \Lambda_c \end{bmatrix}; \quad \hat{\mathbf{M}} = \begin{bmatrix} \mathbf{M}_j & \mathbf{M}_{jc} \\ \mathbf{M}_{jc}^{\mathrm{T}} & \mathbf{I}_{cc} \end{bmatrix}. \tag{7}$$

Related to the displacement vector $\begin{Bmatrix} \mathbf{y}_j \\ \mathbf{c} \end{Bmatrix}$.

$\mathbf{K}_j$ and $\mathbf{M}_j$ denote the residual matrices of the structure statically condensed to the interface dof's y_j by

$$\mathbf{K}_j = \mathbf{K}_{jj} + \mathbf{K}_{ji}\mathbf{G}_{ij},$$

$$\mathbf{M}_j = \mathbf{M}_{jj} + \mathbf{M}_{ji}\mathbf{G}_{ij} + \mathbf{G}_{ij}^{\mathrm{T}}\mathbf{M}_{ji} + \mathbf{G}_{ij}^{\mathrm{T}}\mathbf{M}_{ii}\mathbf{G}_{ij}.$$

The (n_j, n_c) matrix

$$\mathbf{M}_{jc} = \mathbf{G}_{ij}^{\mathrm{T}}\mathbf{M}_{ji}\Phi + \mathbf{M}_{ji}\Phi$$

represents the participation matrix which couples the modal and the interface dof's.

Eq. (6) represents the classical CBT frequently used for substructure coupling analysis. Its accuracy is governed by the number n_c of the normal modes.

3. CMS Method combined with design procedure

The question we are concerned with here is: what happens if a design variable (e.g. a plate thickness or beam property) of a superelement is modified?

In order to save the cost of calculating a new transformation matrix, we might consider using the same transformation (evaluated on the basis of a nominal model) on the modified global matrices of the substructure.

If we consider structural modifications, the modified global matrices become $\mathbf{M} = \mathbf{M}_0 + \Delta\mathrm{M}$ and $\mathbf{K} = \mathbf{K}_0 + \Delta\mathbf{K}$.

Hence, the reduced superelement matrices of modified structure are expressed as $\Delta\hat{\mathbf{K}} = \mathbf{T}_0^{\mathrm{T}}\mathbf{K}\mathbf{T}_0$ and $\Delta\hat{\mathbf{M}} = \mathbf{T}_0^{\mathrm{T}}\mathbf{M}\mathbf{T}_0$, where $\mathbf{T}_0$ is an initial standard CMS transformation matrix.

In the case of global modifications and only in this case, the standard CMS transformation is robust and do not need to be improved. When we talk about global a modification that means the whole substructure is modified by using the same level of perturbation. In this case there is only a frequency shift between the nominal model and the perturbed model, so the nominal transformation can represent correctly the perturbed model.

But when we introduce local modifications, experience shows that the precision of this solution degenerates rapidly with the amplitude of the perturbation. We propose an alternative solution that allows the transformation matrix to be completed with a new basis of vectors which are optimally chosen with respect to the design variables to be modified.

For example, the reduced stiffness matrix $\Delta\hat{K}$ in the case of the Craig-Bampton method is expressed as

$$\Delta\hat{\mathbf{K}} = \mathbf{T}_0^{\mathrm{T}}(\mathbf{K}_0 + \Delta\mathbf{K})\mathbf{T}_0 = \mathbf{T}_0^{\mathrm{T}}\mathbf{K}_0\mathbf{T}_0 + \mathbf{T}_0^{\mathrm{T}}\Delta\mathbf{K}_0\mathbf{T}_0. \tag{8}$$

In this sum, the firts term corresponds to Eq. (6) and is obtained by

$$\mathbf{T}_0^{\mathrm{T}}\mathbf{K}_0\mathbf{T}_0 = \begin{bmatrix} \mathbf{I}_{jj} & \mathbf{0} \\ \mathbf{G}_{ij} & \Phi \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \mathbf{K}_{jj} & \mathbf{K}_{ji} \\ \mathbf{K}_{ji}^{T} & \mathbf{K}_{ii} \end{bmatrix} \begin{bmatrix} \mathbf{I}_{jj} & \mathbf{0} \\ \mathbf{G}_{ij} & \Phi \end{bmatrix} = \begin{bmatrix} \mathbf{K}_j & \mathbf{0} \\ \mathbf{0} & \mathbf{\Lambda}_q \end{bmatrix} \tag{9}$$

and the second term corresponds to the modified stiffness matrix which leads to

$$\begin{aligned} \mathbf{T}_0^{\mathrm{T}}\Delta\mathbf{K}_0\mathbf{T}_0 &= \begin{bmatrix} \mathbf{I}_{jj} & \mathbf{0} \\ \mathbf{G}_{ij} & \Phi \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \Delta\mathbf{K}_{jj} & \Delta\mathbf{K}_{ji} \\ \Delta\mathbf{K}_{ji}^{T} & \Delta\mathbf{K}_{ii} \end{bmatrix} \begin{bmatrix} \mathbf{I}_{jj} & \mathbf{0} \\ \mathbf{G}_{ij} & \Phi \end{bmatrix} \\ &= \begin{bmatrix} \Delta\mathbf{K}_{jj} + \Delta\mathbf{K}_{ji}\mathbf{G}_{ij} + \mathbf{G}_{ij}^{\mathrm{T}}\left(\Delta\mathbf{K}_{ij} + \Delta\mathbf{K}_{ii}\mathbf{G}_{ij}\right) & \mathrm{sym} \\ \Phi^{\mathrm{T}}\left(\Delta\mathbf{K}_{ij} + \Delta\mathbf{K}_{ii}\mathbf{G}_{ij}\right) & \Phi^{\mathrm{T}}\Delta K_{ii}\Phi \end{bmatrix}. \end{aligned} \tag{10}$$

Hence, the reduced stiffness matrix takes the following form

$$\Delta\hat{\mathbf{K}} = \begin{bmatrix} \mathbf{K}_j + \Delta\mathbf{K}_{jj} + \Delta\mathbf{K}_{ji}\mathbf{G}_{ij} + \mathbf{G}_{ij}^{\mathrm{T}}\left(\Delta\mathbf{K}_{ij} + \Delta\mathbf{K}_{ii}\mathbf{G}_{ij}\right) & \mathrm{sym} \\ \Phi^{\mathrm{T}}\left(\Delta\mathbf{K}_{ij} + \Delta\mathbf{K}_{ii}\mathbf{G}_{ij}\right) & \Lambda_q + \Phi^{\mathrm{T}}\Delta\mathbf{K}_{ii}\Phi \end{bmatrix}. \tag{11}$$

Also, the reduced mass matrix is written

$$\Delta\hat{\mathbf{M}} = \mathbf{T}_0^{\mathrm{T}}(\mathbf{M}_0 + \Delta\mathbf{M})\mathbf{T}_0 = \mathbf{T}_0^{\mathrm{T}}\mathbf{M}_0\mathbf{T}_0 + \mathbf{T}_0^{\mathrm{T}}\Delta\mathbf{M}_0\mathbf{T}_0 \tag{12}$$

which takes the following final form:

$$\Delta\hat{\mathbf{M}} = \begin{bmatrix} \mathbf{M}_j + \Delta\mathbf{M}_{jj} + \Delta\mathbf{M}_{ji}\mathbf{G}_{ij} + \mathbf{G}_{ij}^{\mathrm{T}}\left(\Delta\mathbf{M}_{ij} + \Delta\mathbf{M}_{ii}\mathbf{G}_{ij}\right) & \text{sym} \\ \Phi^{\mathrm{T}}\left(\mathbf{M}_{ij} + \Delta\mathbf{M}_{ij}\right) + \Phi^{\mathrm{T}}(\mathbf{M}_{ii} + \Delta\mathbf{M}_{ii})\mathbf{G}_{ij}I_q + \Phi^{\mathrm{T}}\Delta\mathbf{M}_{ii}\Phi & \end{bmatrix}. \tag{13}$$

3.1. Optimal static residual vectors

The strategy for improving the robustness of the reduction transformation with respect to structural modifications will be briefly reviewed in this section. The novelty of this approach lies in its use of a priori knowledge of the nature and localization of the potential design modifications while retaining the amplitude of the modification as a variable. The objective is to construct an optimal set of Ritz vectors, using the nominal model, to complete the standard reduction transformation.

3.1.1. Candidate design modifications

We define a parametric correction of the impedance $\mathbf{Z}$ of the substructure model as follows:

$$\Delta\mathbf{Z} = \Delta\mathbf{K} - \omega^2\Delta\mathbf{M}. \tag{14}$$

By introducing notion of zones or groups of finite elements, it can be written as follows

$$\Delta\mathbf{Z} = \sum_{i=1}^{np}\left[\mathbf{K}_i^{\mathrm{zone}}\left(\Delta p_i\right) - \omega^2\mathbf{M}_i^{\mathrm{zone}}\left(\Delta p_i\right)\right], \tag{15}$$

where Δp_i is the variation of the np design parameter p_i.

In general, the correction matrix is a nonlinear function of the design parameters p_i and, for example, the stiffness modification of a zone can be expressed as

$$\Delta\mathbf{K}_i^{\mathrm{zone}}\left(\Delta p_i\right) = \frac{\Delta p_i}{p_i}\sum_{\alpha}\alpha\left(p_i\right)^{\alpha}\mathbf{K}_{i\alpha}^{\mathrm{zone}}. \tag{16}$$

The homogeneous equation of motion of the modified substructure is given by :

$$[\mathbf{Z}(\omega) + \Delta\mathbf{Z}(\omega)]\mathbf{y}(\omega) = 0. \tag{17}$$

3.1.2. Force basis associated to a structural modification

By introducing the notion of force $\mathbf{f}_\Delta(\omega)$ associated with structural modifications $\Delta\mathbf{Z}(\omega)$, Eq. (17) can be rewritten:

$$\mathbf{Z}(\omega)\mathbf{y}(\omega) = \mathbf{f}_\Delta(\omega), \tag{18}$$

where:

$$\mathbf{f}_\Delta(\omega) = -\mathbf{Z}(\omega)\mathbf{y}(\omega). \tag{19}$$

The vector $\mathbf{f}_\Delta(\omega)$ represents the forces acting on the nominal structure due to the structural modifications. Given that the response vector $\mathbf{y}$ is unknown, the force vector cannot be determined exactly either. The essential idea of the methodology is expressed in the following steps:

1. Eq. (19) is used to generate a basis of force vectors that, if it does not contain the exact force vector with respect to a specific design modification, will at least represent a space containing these vectors. This is accomplished by injecting known nominal model response vectors into Eq. (19).
2. The resulting force basis is then used to generate static response vectors, once again on the basis of the nominal model.
3. The first two steps are repeated for each candidate design parameter.

In practice, many different types of model response vectors may be injected into Eq. (19), including global substructure modes or sensitivity vectors, slave system modes, static vectors, etc. These forces are obviously frequency dependent and are generally evaluated at the frequency corresponding to the response vector in question. Let $\mathbf{B}$ denote the basis of response vectors used for this purpose, hence

$$\mathbf{y}(\omega) = \mathbf{B}\mathbf{c}(\omega). \tag{20}$$

For a modification of the design parameter p_i, Eq. (19) becomes

$$\mathbf{F}^i_\Delta = \left[-\Delta\mathbf{Z}^i(\omega_1)\mathbf{b}_1 \ldots - \Delta\mathbf{Z}^i(\omega_m)\mathbf{b}_m\right]. \tag{21}$$

The final force basis, representative of the set of global set of candidate structural modifications, is then obtained by concatenation of the series of bases generated for each zone

$$\mathbf{F}_\Delta = \left\lfloor \mathbf{F}^1_\Delta \ \mathbf{F}^2_\Delta \ldots \mathbf{F}^{np}_\Delta \right\rfloor. \tag{22}$$

3.1.3. Enriched transformation matrix

Once we have built our force basis, we can compute a series of response vectors, which will be used to complete the standard model reduction transformation matrix. These responses are obviously frequency dependent and in some cases it is worthwhile to calculate them explicitly. However, in order to avoid re-decomposing the global stiffness matrix of the substructure, the static solution is often evaluated instead

$$\mathbf{S}_\Delta = \mathbf{K}^{-1}\mathbf{F}_\Delta. \tag{23}$$

Practically speaking, the basis $\mathbf{S}_\Delta$ is rarely of full rank and must be reconditioned before use. The form this reconditioning takes depends on both the contents of both $\mathbf{S}_\Delta$ and $\mathbf{T}_0$ as will be discussed in the next section. Letting $\Delta\mathbf{T}$ be the reconditioned matrix, the improved transformation matrix $\mathbf{T}$ is given by:

$$\mathbf{T} = [\mathbf{T}_0 \ \ \Delta\mathbf{T}], \tag{24}$$

where $\mathbf{T}_0$ is an initial classical transformation matrix.

The transformation matrix $\mathbf{T}$ contains nj + nc + nr columns where nj is the number of junctions dofs, nc is the number of normal modes retained and nr is the number of static residual vectors retained. It is much more predictive to enrich by this way than add more of normal modes at same size. The enriched base contains necessary information to predict the modified model behavior and especially in case of important modifications that introduce the designer in the optimization procedure of preliminary design.

Fig. 1 illustrates a comparison between an optimization procedure using a standard CMS method or a robust CMS method against structural modifications. The use of the CMS standard method necessists an exact recalculation of the Ritz basis $\mathbf{T}_0$; this can leads to prohibitory of calculations costs for complex models. Whereas, the robust CMS method yields only inexpensive reactualization of the static residual vectors due to the parametric modification contained in the sub-basis $\Delta\mathbf{T}$. This approach can be advantageously adapted to several domains of structural dynamics based on iterative method or: model updating, stochastic structural dynamics, nonlinear problems, reliability-based design optimisation, etc.

In the case of the Craig–Bampton reduction, the complementary basis $\Delta\mathbf{T}$ is calculated with respect to the slave system. Hence, the displacements on the junction dof are zero. So the nominal enriched Craig–Bampton superelement has the form:

$$\tilde{\mathbf{K}}_{\text{enriched}} = \begin{bmatrix} \hat{\mathbf{K}}_{\text{standard}} & & \text{sym} \\ 0 & 0 & \mathbf{Q}^{\mathrm{T}}\mathbf{K}\mathbf{Q} \end{bmatrix}; \quad \tilde{\mathbf{M}}_{\text{enriched}} = \begin{bmatrix} \hat{\mathbf{M}}_{\text{standard}} & & \text{sym} \\ 0 & \mathbf{Q}^{\mathrm{T}}\mathbf{M}\Phi & \mathbf{Q}^{\mathrm{T}}\mathbf{M}\Phi \end{bmatrix} \tag{25}$$

and the current enriched superelement model has the following one :

$$\Delta\tilde{\mathbf{K}}_{\text{enriched}} = \begin{bmatrix} \Delta\hat{\mathbf{K}}_{\text{standard}} & & \text{sym} \\ \mathbf{Q}^{\mathrm{T}}\left(\Delta\mathbf{K}_{ij} + \Delta\mathbf{K}_{ii}\mathbf{G}_{ij}\right) & \mathbf{Q}^{\mathrm{T}}\Delta\mathbf{K}\Phi & \mathbf{Q}^{\mathrm{T}}(\mathbf{K}+\Delta\mathbf{K})\mathbf{Q} \end{bmatrix};$$

$$\Delta\tilde{\mathbf{M}}_{\text{enriched}} = \begin{bmatrix} \Delta\hat{\mathbf{M}}_{\text{standard}} & & \text{sym} \\ \mathbf{Q}^{\mathrm{T}}\left(\mathbf{M}_{ij} + \Delta\mathbf{M}_{ij}\right)\mathbf{I}_{jj} + \mathbf{Q}^{\mathrm{T}}(\mathbf{M}_{ii} + \Delta\mathbf{M}_{ii})\mathbf{G}_{ij} & \mathbf{Q}^{\mathrm{T}}\Delta\mathbf{M}\Phi & \mathbf{Q}^{\mathrm{T}}(\mathbf{M}+\Delta\mathbf{M})\mathbf{Q} \end{bmatrix}. \tag{26}$$

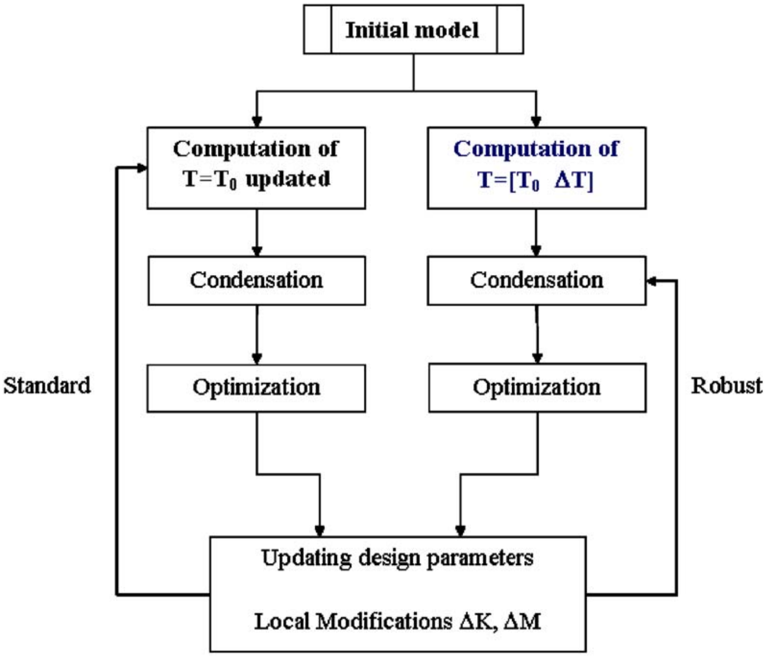

Fig. 1. Optimization procedure: standard CMS vs. robust CMS method.

Remarks

- The static responses $\mathbf{S}_\Delta$ are often linearly dependent and it is important to perform an adequate reconditioning of this matrix before use. The form of this pre-processing depends on the contents of $\mathbf{T}_0$. For example, if part or all of $\mathbf{T}_0$ is the modal matrix of the substructure, then it is advantageous to replace $\mathbf{S}_\Delta$ by the corresponding static residual matrix formed by removing the static contribution of the retained modes.
- When possible, it is important not to combine the contributions of several zones to construct a single (sub) basis. This has the effect of coupling otherwise independent parameters.

3.1.4. Reduction of the static residual coordinates

As we can see in Eq. (26) the enriched superelement is larger than the standard superelement. The size depends on the number of static residual vectors we retain in the transformation basis. In order to avoid getting a too much large superelement, we can eliminate the generalized coordinates associated with the static residual vectors without loosing in accuracy.

The eigenequation of motion for the coupled perturbed system is expressed as:

$$\left(\begin{bmatrix} \Delta\mathbf{K}_{\mathrm{cms}} & \Delta\mathbf{K}_{\mathrm{cms},r} \\ \Delta\mathbf{K}^{\mathrm{T}}_{\mathrm{cms},r} & \Delta\mathbf{K}_{rr} \end{bmatrix} - \omega^2 \begin{bmatrix} \Delta\mathbf{M}_{\mathrm{cms}} & \Delta\mathbf{M}_{\mathrm{cms},r} \\ \Delta\mathbf{M}^{\mathrm{T}}_{\mathrm{cms},r} & \Delta\mathbf{M}_{rr} \end{bmatrix}\right) \begin{Bmatrix} \mathbf{q}_{\mathrm{cms}} \\ \mathbf{r} \end{Bmatrix} = \{\mathbf{f}\}, \tag{27}$$

where $\mathbf{q}_{\mathrm{cms}}$ correspond to the standard coordinates of any CMS method.

To eliminate the generalized coordinates r, we assume in Eq. (27) that the effect of the residual mass matrices $\Delta\mathbf{M}_{\mathrm{cms},r}$ and $\Delta\mathbf{M}_{rr}$ can be neglected.

This hypothesis may not be valid if there is only a mass modification but in general, the stiffness dominates in low-frequency range and anyway, the enriched reduced system is to be used in updating or optimization procedure. Physically, in these procedures there is generally one or many parameters which modify the stiffness and the eigenmodes for which the firsth modes of a structure are concerned for the procedure.

So if we neglect the effect of residual mass matrices $\Delta\mathbf{M}_{\text{cms},r}$ and $\Delta\mathbf{M}_{rr}$, the reduced modified problem becomes

$$\left(\begin{bmatrix}\Delta\mathbf{K}_{\text{cms}} & \Delta\mathbf{K}_{\text{cms},r}\\ \Delta\mathbf{K}^{\text{T}}_{\text{cms},r} & \Delta\mathbf{K}_{rr}\end{bmatrix} - \omega^2\begin{bmatrix}\Delta\mathbf{M}_{\text{cms}} & 0\\ 0 & 0\end{bmatrix}\right)\begin{Bmatrix}\mathbf{q}_{\text{cms}}\\ \mathbf{r}\end{Bmatrix} = 0. \tag{28}$$

Let us eliminate the generalized coordinates r using the second row in Eq. (28)

$$\{\mathbf{r}\} = \mathbf{G}_{\text{cms,r}}\{\mathbf{q}_{cb}\}, \tag{29}$$

where $\mathbf{G}_{\text{cms},r} = -\Delta\mathbf{K}_{rr}^{-1}\Delta\mathbf{K}^{\text{T}}_{\text{cms},r}$.

By substituting Eq. (29) Into Eq. (27) and premultiplying by $\begin{bmatrix}\mathbf{I}_{cb}\\ \mathbf{G}_{\text{cms},r}\end{bmatrix}^{\text{T}}$ we obtain the following new eigenequation of motion for the coupled perturbed as defined by the form:

$$\begin{aligned}&\Big(\Big[\Delta\mathbf{K}_{\text{cms}} + \Delta\mathbf{K}_{\text{cms},r}\mathbf{G}_{\text{cms},r} + \mathbf{G}^{T}_{\text{cms},r}\Delta\mathbf{K}^{\text{T}}_{\text{cms},r} + \mathbf{G}^{\text{T}}_{\text{cms},r}\Delta\mathbf{K}_{rr}\mathbf{G}_{\text{cms},r}\Big]\\ &\quad -\boldsymbol{\omega}^2\Big[\Delta\mathbf{M}_{\text{cms}} + \Delta\mathbf{M}_{\text{cms},r}\mathbf{G}_{\text{cms},r} + \mathbf{G}^{\text{T}}_{\text{cms},r}\Delta\mathbf{M}^{\text{T}}_{\text{cms},r} + \mathbf{G}^{\text{T}}_{\text{cms},r}\Delta\mathbf{M}_{rr}\mathbf{G}_{\text{cms},r}\Big]\Big)\{\mathbf{q}_{\text{cms}}\} = \{\tilde{\mathbf{f}}\}.\end{aligned} \tag{30}$$

The resulting condensed model is the same size of a standard Craig–Bampton model but it is more robust and can be used in updating or optimization procedure. Moreover the computation to obtain an enriched condensed model is much less costly than an exact computation from the whole structure.

4. Numerical results

4.1. Academic example

Consider a simple structure consisting of a plate system shown in Fig. 2. The proposed methodology will be illustrated on a simple example of plates comprising two Craig–Bampton superelements and a residual structure containing, by definition, the set of junction dof for all superelements as well as physical dof corresponding to unreduced structural elements.

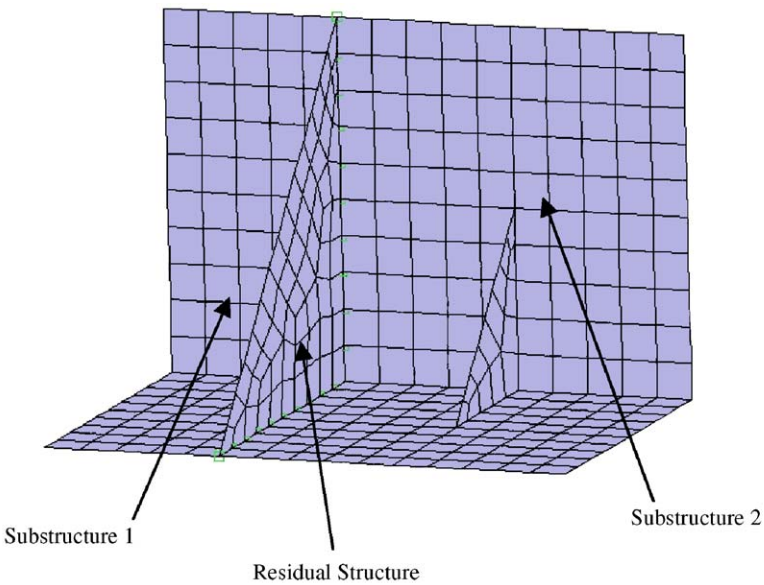

Fig. 2. Finite element model of the academic example.

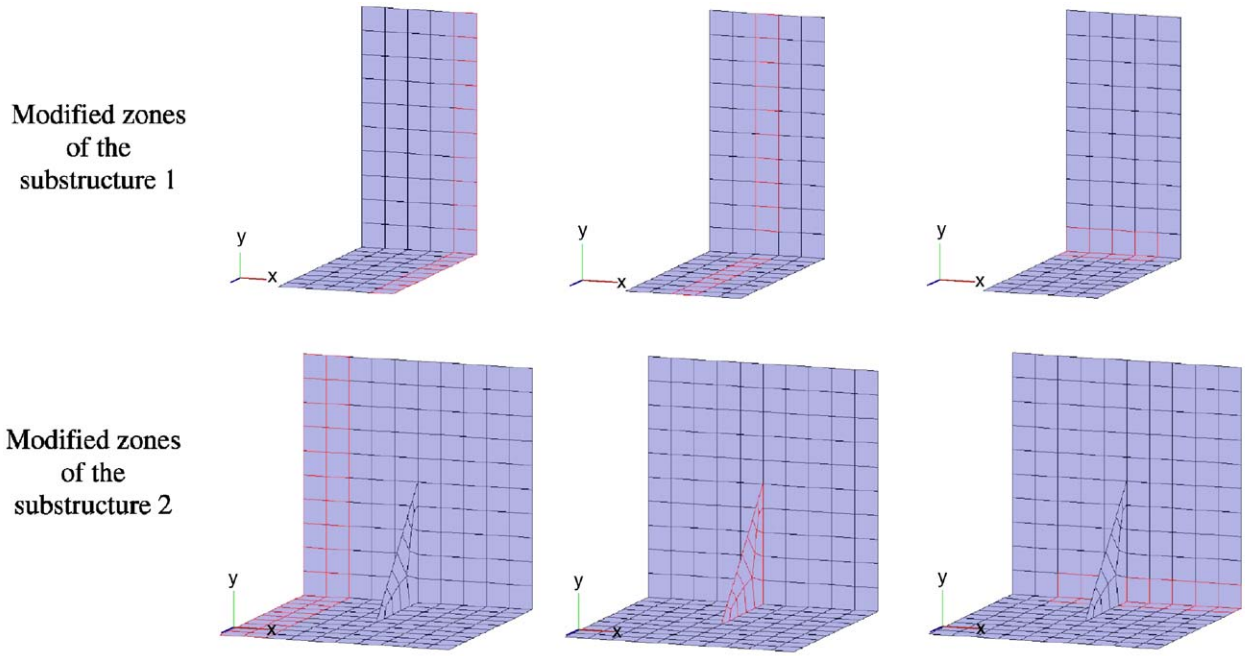

Fig. 3. Definition of the modified zones of the components.

Table 1
Composition of the CBT, CBE and CBEE reduction

		CBT reduction		CBE reduction		CBEE reduction	
		Substructure		Substructure		Substructure	
		1	2	1	2	1	20
Number of coordinates associated with	Constraint modes	126	126	126	126	126	126
	Normal modes	30	30	30	30	30	30
	Optimal vectors	0	0	104	85	0	0
Reduced model size		156	156	260	241	156	156

The numerical model contains a total of about 2800 dof and is under free–free boundary conditions. Substructure 1, a square, contains 756 dof with 126 interface dof while substructure 2 contains 1470 dof with 126 interface degrees of freedom. The residual structure contains 462 dofs. The both superelements have been created using the Craig–Bampton technique.

In order to illustrate the proposed methodology, we have perturbed the nominal substructure models as follows:

- Each substructure contains three local zones (Fig. 3) where the Young's moduli and the thickness are modified by 0.25, 1 or 4.

We will compare the accuracy between three types of calculation:

1. Re-using the standard CBT of each of the nominal substructure models on the corresponding perturbed models. This method is denoted by CB method.
2. Enrichment of the CBT by a set of optimally selected Ritz vectors. This method is denoted by CBE method.
3. Enrichment of the standard CBT and elimination of the generalized coordinates associated with the set of optimally selected Ritz vectors. This method is denoted by CBEE method.

In order to demonstrate the improved convergence properties of the optimal basis vectors, the dimensions of the first and third transformation matrices are chosen to be identical. The dimension of the reduction bases for the methods is reported in Table 1.

4.1.1. Comparison criteria

The relative precision of the different analyses will be compared on the basis of three classic criteria: the relative frequency error (ε_f), the mean square error (MSE) [15] and modal assurance criteria (MAC).

$$\mathrm{MSE} = \frac{\sum_{i=1}^{m} (\mathrm{Exact} - \mathrm{Pred})^2}{m\sigma^2_{\mathrm{Exact}}}, \tag{31}$$

where m is the number of experiments conducted, Exact and Pred are the exact and predicted values of the output feature, and σ_{Exact} is the standard deviation of the Exact values of the output feature.

$$\varepsilon_{\mathrm{MAC}} = \frac{\left(\mathrm{y}^{\mathrm{T}}_{\mathrm{exact}}\mathrm{y}_R\right)^2}{\left(\mathrm{y}^{\mathrm{T}}_{\mathrm{exact}}\mathrm{y}_{\mathrm{exact}}\right)\left(\mathrm{y}^{\mathrm{T}}_{\mathrm{R}}\mathrm{y}_R\right)} \times 100, \tag{32}$$

where $\mathbf{y}_{\mathrm{exact}}$ an eigenvector of the assembled perturbed system calculated using an exact update of the modified superelements and $\mathbf{y}_R$ an eigenvector of the assembly system calculated using either the CBT or CBEE transformation on the perturbed mode.

4.2. Results

The 2 first frequencies of the $3^6 = 729$ configurations (6 zones with 3 levels of modification) have been computed using the methods defined before (exact, CBT, CBE and CBEE). To compare the results obtained by substructuring to the exact results, mean square error and the maximum and the mean of the relative frequency error are shown in the Figs. 4, 5 and Table. 2. The dashed line in Fig. 4 is the limit of acceptance of 0.09028 defined before (MSE_max$\approx$0.09). Only both CBE and CBEE methods lead to a MSE below this limit, that mean the frequency family of the 2 firsts modes computed from the CBE and CBEE methods are

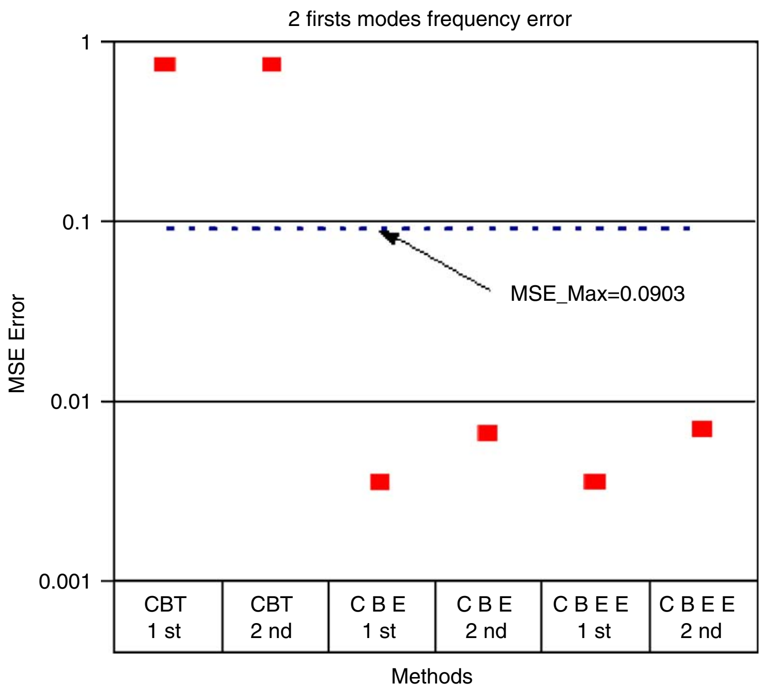

Fig. 4. MSE error of CMS methods CBT, CBE and CBEE (- - -, MSE_Max; ▬, MSE error).

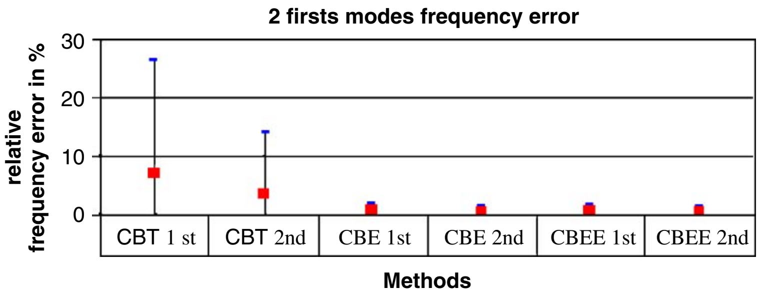

Fig. 5. Frequency error of the CBT, CBE, CBEE methods (▬ ε_f Max%; ■, ε_f Mean%; ε_f Min % $=0$).

Table 2
MSE error and frequency error for the two firsts modes

	CBT 1st	CBT 2nd	CBE 1st	CBE 2nd	CBEE 1st	CBEE 2nd
MSE-Max	0.0903	0.0903	0.0903	0.0903	0.0903	0.0903
MSE	0.7251	0.7234	0.0035	0.0065	0.0035	0.0069
ε_f Max en %	26.47	13.84	1.52	1.06	1.52	1.0718
ε_f Min en %	0	0	0	0	0	0
ε_f Mean en %	6.91	3.37	0.5	0.34	0.5	0.3526

Table 3
Structural modifications level

	Substructure 1			Substructure 2		
	E Zone 1	T Zone 2	E Zone 3	E Zone 4	T Zone 5	E Zone 6
Modification factor	4	0.25	4	4	0.25	4

very close, but it is not the case for the CBT method. Fig. 5 confirms this tendency, indeed the CBE and the CBEE methods lead to a maximum frequency relative error of 1.5% compared to the 26% of the CBT method.

For example, we can study the results of one out of the 729 modification configurations. The configuration chosen is defined in Table 3 and a frequency range of interest [0-1000 Hz] is defined for the study.

It is well known that the local modifications are poorly handled without recalculation of the CBT. Fig. 6 and Table 4 report the relative frequency error and MAC criteria for the first perturbation. As expected, the CBT performs poorly, while the proposed methodology (CBEE) yields a significantly more accurate prediction.

4.3. Industrial application

4.3.1. Description of test case

The proposed methodology will be illustrated on a subassembly of an aero-engine model (Fig. 7) comprising four Craig–Bampton superelements and a residual structure containing, by definition, the set of junction dof for all superelements as well as physical dof corresponding to unreduced structural elements.

The numerical model contains a total of about 40 000 dof and is under free–free boundary conditions. Substructure 1, a casing, contains 9000 dof with 300 interface dof while substructures 2 and 3 are both stators containing respectively 4200 dof with 600 interface degrees of freedom and 4400 dof with 600 interface dof. Substructure 4 contains 6000 dof with 60 interface dof and is also a casing. An analysis range is defined which contains 6 rigid body modes and 12 flexible modes in the initial configuration. Superelements for all four components have been created using the CBT.

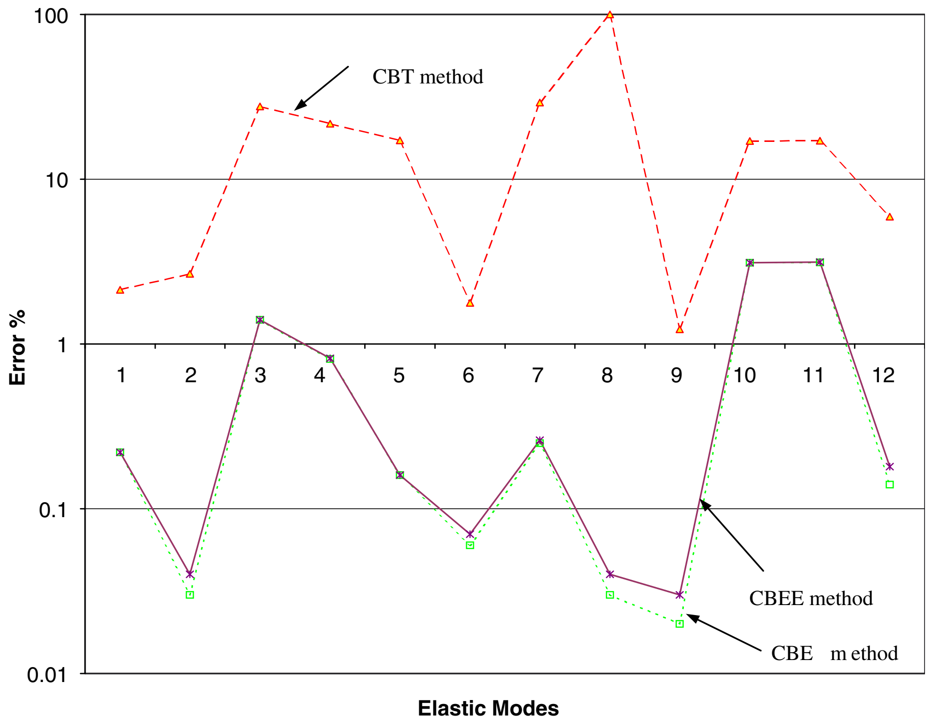

Fig. 6. Relative frequency error associated to the modification no 1 (–Δ–, CBT method; -□-, CBE method; -*-, CBEE method).

In order to illustrate the proposed methodology, we have perturbed the nominal substructure models as follows:

- Concerning the first substructure, we define two zones where the Young's moduli are modified by a factor 2.
- Concerning the second and the third substructures which are two tubes we choose to introduce local modifications. We define two zones for each substructure where the Young's moduli are modified respectively by a factor 2 and 0.5.
- And for the last substructure, we define a zone where the thickness is the modification parameter. The nominal values of the shell element thickness are divided by 2.

These structural perturbations lead to maximum frequency shifts on the order of 20%.

We will compare the precision between two types of calculation:

1. Reusing the standard CBT of each nominal substructure models on the corresponding perturbed models.
2. Enrichment of the standard CBT and elimination of the generalized coordinates associated with the set of optimally selected Ritz vectors. This method is denoted by CBEE method.

In order to demonstrate the improved convergence properties of the optimal basis vectors, the dimensions of the two transformation matrices are chosen to be identical, that is to say, the number of slave eigenmodes included in Φ are increased for the standard CBT. The dimension of the reduction bases for both methods is reported in Tables 5 and 6.

4.3.2. Results

Table 6 and 7 reports the relative frequency error and MAC criteria for the two analyses. We note a general improvement in the precision of all eigenfrequencies and eigenvectors. The remaining modes are local to

Table 4
MAC associated to the modification no 1

CBT	Exact modes											
	1	2	3	4	5	6	7	8	9	10	11	12
1	.99											
2		.99										
3				.89								
4			.89									
5						.92						
6					.79							
7									.98			
8				.14			.79					
9										.96		
10											.86	
11											.15	.80
12												
CBEE	**Exact modes**											
	1	2	3	4	5	6	7	8	9	10	11	12
1	.99											
2		1.0										
3			.99									
4				.99								
5					.99							
6						.99						
7							.99					
8								.98				
9									.99			
10										.99		
11											.99	
12												.99

individual substructures. Fig. 8 shows the exact (nominal and perturbed) and approximate transfer functions for two different collocated observation–excitation points. In both cases, the modal damping coefficient was defined to be 2%. Note that axis labels have been purposely omitted. The dotted and dashed lines represent the exact transfer function of the perturbed and nominal models, respectively, while the dot-dashed and continuous lines represent the standard and enriched Craig–Bampton results, respectively. We can observe that the standard Craig–Bampton transformation largely underestimates the magnitude of the frequency shifts of the perturbed structure while the proposed methodology yields a significantly more accurate prediction.

5. Conclusions

In this paper, we have enriched an existing methodology for the approximate reanalysis of linear elastodynamic behavior to superelement technology. This approach allows design parameters of a superelement to be modified, for example in the context of an optimization algorithm, without having to perform a complete superelement analysis at each point in parameter space.

The method is based on the enrichment of the standard superelement reduction transformation on a basis of representation which is optimized with respect to the design parameters to be modified. The proposed methodology can be integrated in a variety of component mode synthesis techniques and we have illustrated its use in the context of the Craig–Bampton superelement. This approach is a little bit more costly than a standard CMS method in term of computation but definitely more economic than an exact reanalysis. Moreover, this approach is particularly effective when dealing with very large models having a large number of subassemblies. The improved model reduction transformation can be prepared in advance for each

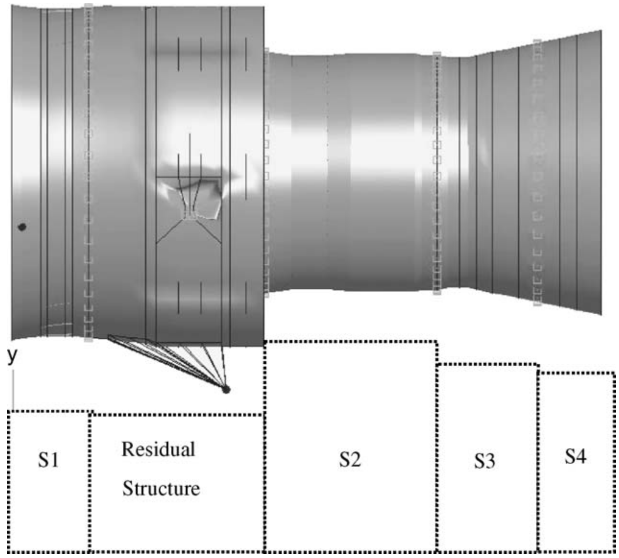

Fig. 7. Aero-engine model.

Table 5
Composition of the standard CBT reduction

	Substructure			
	1	2	3	4
Constraint modes	300	600	600	60
Normal modes	20	40	40	30
Optimal vectors	0	0	0	0
Reduced model size	320	640	640	90

Table 6
Composition of the CBEE reduction

	Substructure			
	1	2	3	4
Constraint modes	300	600	600	60
Normal modes	20	40	40	30
Optimal vectors	5	20	20	10
Elimination of the optimal vectors	−5	−20	−20	−10
Reduced model size	320	640	640	90

substructure and then used directly during the optimization procedure, thus avoiding the exact recalculation of the modified superelements.

An academic example and an industrial test case derived from the aero-engine industry was used to illustrate the efficiency of the optimized basis in representing the modified behavior of a perturbed structure.

Current work is in progress to take into account local nonlinearities. The purpose is to apply the robust reduction method to a system with local nonlinearity to asses its potential for nonlinear dynamics.

Table 7
Comparison of relative prediction errors

Mode number	CBT		CBEE	
	ε_f	ε_{MAC}	ε_f	ε_{MAC}
1	0.64	1.0	0.60	1.0
2	0.63	1.0	0.60	1.0
3	4.30	1.0	0.06	1.0
4	0.03	1.0	0.03	1.0
5	1.40	1.0	1.08	1.0
6	1.76	1.0	1.04	1.0
7	1.05	1.0	0.43	1.0
8	3.28	0.7	1.37	1.0
9	3.57	0.5	0.64	1.0
10	5.07	0.3	1.32	1.0
11	12.82	0.2	2.88	1.0
12	3.82	0.7	2.11	1.0

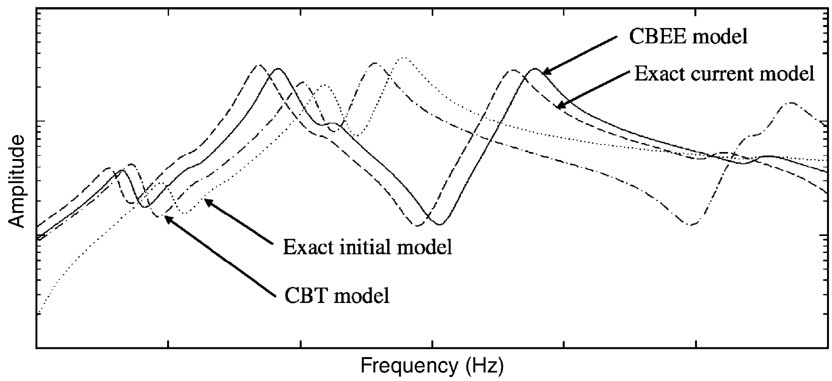

Fig. 8. Prediction of the frequency Response Function—CMS methods : CBT, CBE, CBEE.

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References

[1] E. Balmès, Parametric families of reduced finite element models theory and applications, *Mechanical Systems and Signal Processing* 10 (4) (1996) 381–394.
[2] E. Balmès, Optimal Ritz vectors for component mode synthesis using the singular value decomposition, *AIAA Journal* 34 (1996) 1256–1260.
[3] Ph. Guillaume, M. Masmoudi, Computation of high order derivatives in optimal shape design, *Numerishe Mathematik* 67 (1994) 231–250.
[4] A. Bouazzouni, G. Lallement, S. Cogan, Selecting a Ritz basis for reanalysis of frequency response functions of modified structures, *Journal of Sound and Vibration* 199 (2) (1997) 309–322.
[5] S. Grihon, M. Mahé, F. Seigneuret, Une méthode de réduction de modèle pour la réanalyse modale et aéroélastique de structure d'avion, *4ème colloque en calcul des structures* 1999, pp. 173–178.
[6] R.R. Craig Jr., M.C.C. Bampton, Coupling of substructures for dynamic analysis, *AIAA Journal* 6 (1968) 1313–1319.
[7] W.C. Hurty, Dynamic analysis of structural systems using component modes, *AIAA Journal* 3 (1965) 678–685.

[8] R.H. MacNeal, A hybrid method of component mode synthesis, *Computers and Structures* 1 (4) (1971) 581–601.
[9] S. Rubin, Improved component mode representation for structural dynamic analysis, *AIAA Journal* 13 (8) (1975) 995–1006.
[10] R.L. Goldman, Vibration analysis by dynamic partitioning, *AIAA Journal* 7 (1969) 1152–1154.
[11] S.N. Hou, Review of modal synthesis techniques and a new approach, *Shock and Vibration Bulletin* 40 (4) (1969) 25–30.
[12] D.R. Martinez, A.K. Miller, T.G. Carne, Combined experimental/analatycal modeling of shell/payload structures, in: D.R. Martinez, A.K. Miller (Eds.), *Combined Experimental/Analatycal Modeling of Dynamic Structural Systems AMD ASME*, Vol. 67, 1985, pp. 167–184.
[13] J.P. Lombard, *Contribution to finite elements models reduction by component mode synthesis*, Thèse de doctorat en sciences pour l'ingenieur, Université de Franche-Comté, 1999.
[14] W.A. Benfield, R.F. Hrurda, Vibration analysis of structures by component mode substitution, *AIAA Journal* 9 (1971) 1255–1261.
[15] G.J. Battaglia, J.M. Maynard Mean square error: a useful tool for statistical process management, *AMP Journal of Technology* 2 (1992) 47–55.